L-Systems and Affine Transformations

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Introduction

- We have seen that fractals are very effective for squeezing a great deal of length (perhaps infinite) into a finite area.
- It is not surprising that fractals are used within living organisms for achieving special goals:
  - Human brain (maximize surface area)
  - Respiratory system, circulatory system (tree-like branching)
Production systems

• The “instructions” for building such complex structures must be stored somewhere (e.g., within the DNA)
• It is therefore important to be able to store that information as compactly as possible
• **L-Systems** were proposed in 1968 by biologist Arstid Lindenmayer as a mathematical description of how plants grow
  – Can be applied to other structures as well
L-Systems

- An L-System consists of a special “seed” cell (axiom), and a set of rules describing how new cells can be generated from old cells.

- Example:

<table>
<thead>
<tr>
<th>Axiom</th>
<th>B</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rules</td>
<td>B → F[-B] + B</td>
</tr>
<tr>
<td></td>
<td>F → FF</td>
</tr>
</tbody>
</table>

  expands to

  B → F[-B] + B
  → ...

  ...
Turtle graphics commands

- **F** Draw forward by a fixed length
- **G** Go forward by a fixed length (without drawing)
- **+** Turn right by a fixed angle ($n+$ turns right $n$ times)
- **-** Turn left by a fixed angle ($n-$ turns left $n$ times)
- **[** Save the current position in the stack
- **]** Restore position from the top of the stack
- **|** Draw forward by a length computed from the execution depth
Koch Curve

- Angle: 60
- Axiom: F
- Rule: F → F-F++F-F

```
lsys -da 60 -axiom "F" -rule "F=F-F++F-F"
```
Koch Snowflake

- Angle: 60
- Axiom: F++F++F
- Rule: F → F-F++F-F

```
lsys -da 60 -axiom "F++F++F" -rule "F=F-F++F-F"
```
Peano curve

- Angle: 90
- Axiom: F
- Rule: $F \rightarrow F-F+F+F+F-F-F-F+F$

```bash
lsys -da 90 -axiom "F" -rule "F=F-F+F+F+F-F-F-F+F"
```
### Other examples

<table>
<thead>
<tr>
<th>Tree 1</th>
<th>Angle: 90</th>
<th>Axiom: $F \rightarrow \ldots$</th>
<th>Rule(s): $F = [\ldots F \ldots F]$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Tree 2</td>
<td>Angle: 90</td>
<td>Axiom: $F$</td>
<td>Rule(s): $F = [\ldots F \ldots F]$</td>
</tr>
<tr>
<td>Tree 3</td>
<td>Angle: 90</td>
<td>Axiom: $F$</td>
<td>Rule(s): $F = [\ldots F \ldots F]$</td>
</tr>
<tr>
<td>Tree 4</td>
<td>Angle: 90</td>
<td>Axiom: $F$</td>
<td>Rule(s): $F = [\ldots F \ldots F]$</td>
</tr>
<tr>
<td>Tree 5</td>
<td>Angle: 90</td>
<td>Axiom: $F$</td>
<td>Rule(s): $F = [\ldots F \ldots F]$</td>
</tr>
</tbody>
</table>
Affine Transforms

- Affine transformations can be used to easily describe fractal objects that are made of “smaller copies of themselves”
  - Possibly scaled and/or translated and/or rotated
Translation

\[
\begin{pmatrix}
x' \\
y'
\end{pmatrix} = \begin{pmatrix}
x \\
y
\end{pmatrix} + \begin{pmatrix}
\Delta x \\
\Delta y
\end{pmatrix}
\]
Scaling

\[
\begin{pmatrix}
  x' \\
  y'
\end{pmatrix}
= \begin{pmatrix}
  sx & 0 \\
  0 & sy
\end{pmatrix}
\begin{pmatrix}
  x \\
  y
\end{pmatrix}
\]
Reflection

\[
\begin{pmatrix}
x' \\
y'
\end{pmatrix} = \begin{pmatrix}
-1 & 0 \\ 0 & 1
\end{pmatrix}\begin{pmatrix}
x \\
y
\end{pmatrix}
\]

\[
\begin{pmatrix}
x' \\
y'
\end{pmatrix} = \begin{pmatrix}
1 & 0 \\ 0 & -1
\end{pmatrix}\begin{pmatrix}
x \\
y
\end{pmatrix}
\]
Rotation

\[
\begin{pmatrix}
  x' \\
  y'
\end{pmatrix}
= \begin{pmatrix}
  \cos(\theta) & -\sin(\theta) \\
  \sin(\theta) & \cos(\theta)
\end{pmatrix}
\begin{pmatrix}
  x \\
  y
\end{pmatrix}
\]
Composing linear transformations

- Rotation, scaling, flipping + translation

\[
\begin{pmatrix}
  x' \\
  y'
\end{pmatrix} = \begin{pmatrix}
  a & b \\
  c & d
\end{pmatrix} \begin{pmatrix}
  x \\
  y
\end{pmatrix} + \begin{pmatrix}
  e \\
  f
\end{pmatrix}
\]

- With a suitable choice for terms $a$ through $f$, this can be rewritten using a single matrix product

\[
\begin{pmatrix}
  x' \\
  y'
\end{pmatrix} = \begin{pmatrix}
  a & b & c \\
  d & e & f \\
  0 & 0 & 1
\end{pmatrix} \begin{pmatrix}
  x \\
  y \\
  1
\end{pmatrix}
\]

- To compose multiple transformations: $x' = C \left( B \left( A x \right) \right)$
The Multiple Copy Reduction Machine (MCRM)

A

One or more affine linear transformations

A A A

Seed Image

Recursion

Next Image
The Multiple Copy Reduction Machine (MCRM)

Source: *Il linguaggio dei frattali*, Le Scienze n. 266, october 1990
Example
Speeding up convergence

• Start with a random point
• Iterate the point for n steps (n large)
  - At each step choose one transformation with probability proportional to the area of the transformed box, i.e., the determinant of matrix

\[
\begin{pmatrix}
  a & b \\
  c & d
\end{pmatrix}
\]

• Skip the first iterations, plot the positions of all other points
The Multiple Copy Reduction Machine (MCRM)

\[
\begin{pmatrix} x \\ y \end{pmatrix} \xrightarrow{\text{Random choice}} \begin{pmatrix} x' \\ y' \end{pmatrix}
\]

One transformation

One transformation

One transformation

Recursion

Next point
Examples
# Examples

<table>
<thead>
<tr>
<th>$a.$</th>
<th>$b.$</th>
<th>$c.$</th>
<th>$d.$</th>
<th>$e.$</th>
<th>$f.$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.7500</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.7500</td>
<td>0.1250</td>
<td>0.1250</td>
</tr>
<tr>
<td>0.5000</td>
<td>-0.5000</td>
<td>0.5000</td>
<td>0.5000</td>
<td>0.5000</td>
<td>0.0000</td>
</tr>
<tr>
<td>0.2500</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.2500</td>
<td>0.0000</td>
<td>0.7500</td>
</tr>
<tr>
<td>0.2500</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.2500</td>
<td>0.7500</td>
<td>0.7500</td>
</tr>
<tr>
<td>0.2500</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.2500</td>
<td>0.0000</td>
<td>0.0000</td>
</tr>
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<td>0.0000</td>
<td>0.0000</td>
<td>0.2500</td>
<td>0.7500</td>
<td>0.0000</td>
</tr>
</tbody>
</table>

![Diagram of a fractal pattern](image)
Examples

<p>| | | | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>a.</td>
<td>b.</td>
<td>c.</td>
<td>d.</td>
<td>e.</td>
<td>f.</td>
</tr>
<tr>
<td>0.1950</td>
<td>-0.4880</td>
<td>0.3440</td>
<td>0.4430</td>
<td>0.4431</td>
<td>0.2453</td>
</tr>
<tr>
<td>0.4620</td>
<td>0.4140</td>
<td>-0.2520</td>
<td>0.3610</td>
<td>0.2511</td>
<td>0.5692</td>
</tr>
<tr>
<td>-0.0580</td>
<td>-0.0700</td>
<td>0.4530</td>
<td>-0.1110</td>
<td>0.5976</td>
<td>0.0969</td>
</tr>
<tr>
<td>-0.0350</td>
<td>0.0700</td>
<td>-0.4690</td>
<td>-0.0220</td>
<td>0.4884</td>
<td>0.5069</td>
</tr>
<tr>
<td>-0.6370</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.5010</td>
<td>0.8562</td>
<td>0.2513</td>
</tr>
</tbody>
</table>
# Examples

<table>
<thead>
<tr>
<th></th>
<th>a</th>
<th>b</th>
<th>c</th>
<th>d</th>
<th>e</th>
<th>f</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.5000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.7500</td>
<td>0.2500</td>
<td>0.0000</td>
</tr>
<tr>
<td>2</td>
<td>0.2500</td>
<td>-0.2000</td>
<td>0.1000</td>
<td>0.3000</td>
<td>0.2500</td>
<td>0.5000</td>
</tr>
<tr>
<td>3</td>
<td>0.2500</td>
<td>0.2000</td>
<td>-0.1000</td>
<td>0.3000</td>
<td>0.5000</td>
<td>0.4000</td>
</tr>
<tr>
<td>4</td>
<td>0.2000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.3000</td>
<td>0.4000</td>
<td>0.5500</td>
</tr>
</tbody>
</table>