Fractals

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Geometric Objects

- Man-made objects are geometrically simple (e.g., rectangles, circles, triangles...).
  - Those simple objects seem appropriate for describing many artifacts (e.g., this screen is roughly rectangular, tires are circular, power lines are long lines, etc.
- What about natural objects? (snowflake, tree, mountain, clouds...)
  - These objects have been considered exceptions to the rule

Fractals

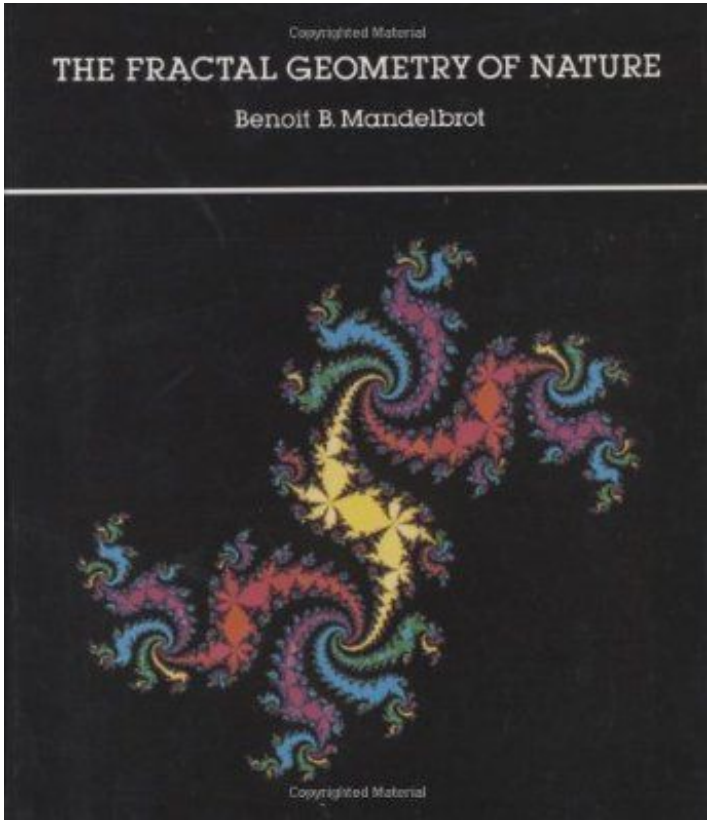
- The term fractal was coined by the mathematician Benoit B. Mandelbrot to differentiate geometric figures with other figures that can not be easily classified in conventional terms

Clouds are not spheres, mountains are not cones, coastlines are not circles, and bark is not smooth, nor does lightning travel in a straight line.

—Mandelbrot, in his introduction to *The Fractal Geometry of Nature*

Benoît B. Mandelbrot (1924 – 2010)

Fractals



Benoit B. Mandelbrot, *The Fractal Geometry of Nature,* W. H. Freeman and Company, ISBN 978-0716711865

Benoit B. Mandelbrot, *Gli Oggetti Frattali*, Einaudi, 2000, ISBN 9788806155667

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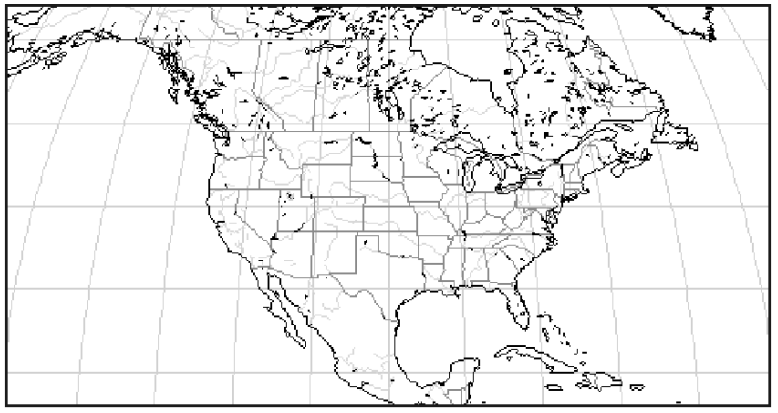
Fractals

- Self-similarity
  - Fractals are self-similar at several scales, meaning that a small portion of a fractal looks similar to the whole object
- Fractional dimension
  - Fractals have fractional dimension: a fractal object with dimension 1.5 is somewhat more than a line (dimension 1) and less than a plane (dimension 2)



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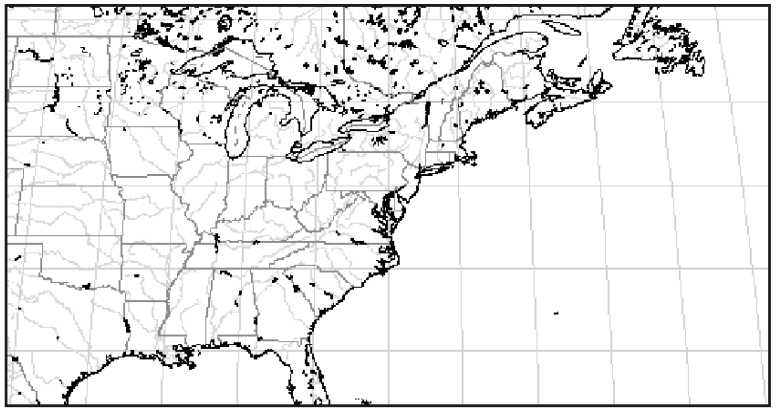

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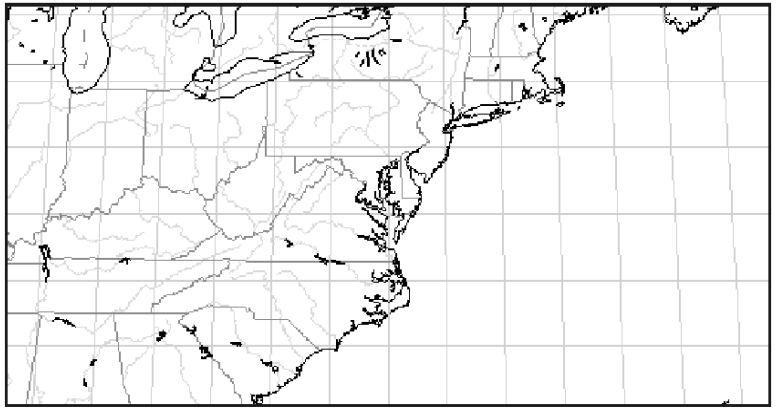

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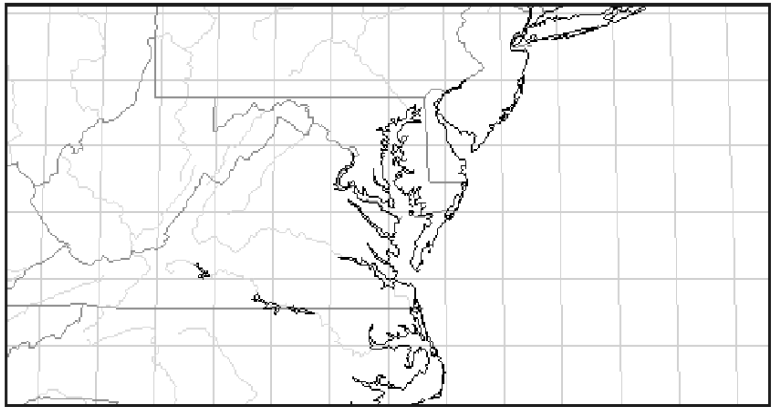

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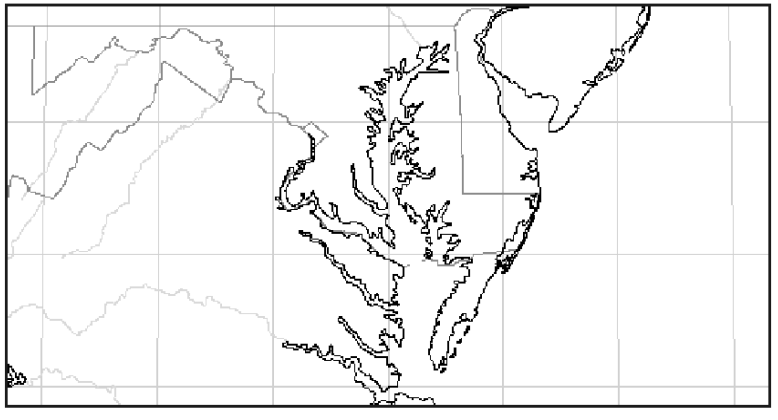
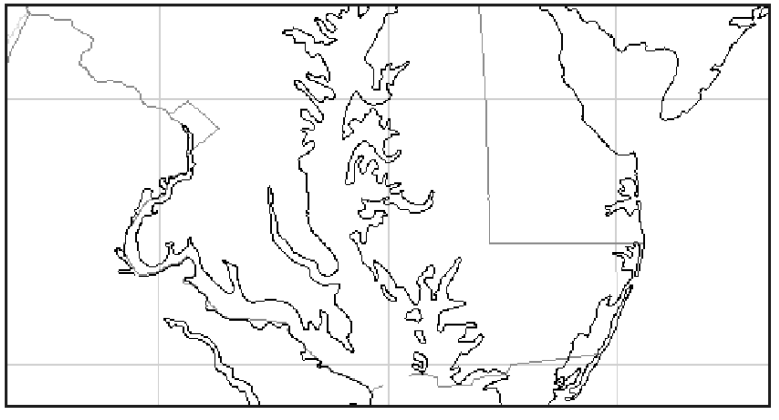




The Cantor Set

- Start with the interval [0,1]
- Remove the middle third: [0,1/3] U [2/3,1]
- Remove the middle third for each interval:
  [0, 1/9] U [2/9, 1/3] U [2/3, 7/9] U [8/9, 1]
- Iterate the procedure *forever*

Measure of the Cantor set

- We want to measure the “width” of the Cantor set
- At step n (starting from $n = 0$) the Cantor set contains 2^n intervals of width $1/3^n$ each; total width is $(2/3)^n$
- For $n \to +\infty$ the total width becomes 0

Points in the Cantor set

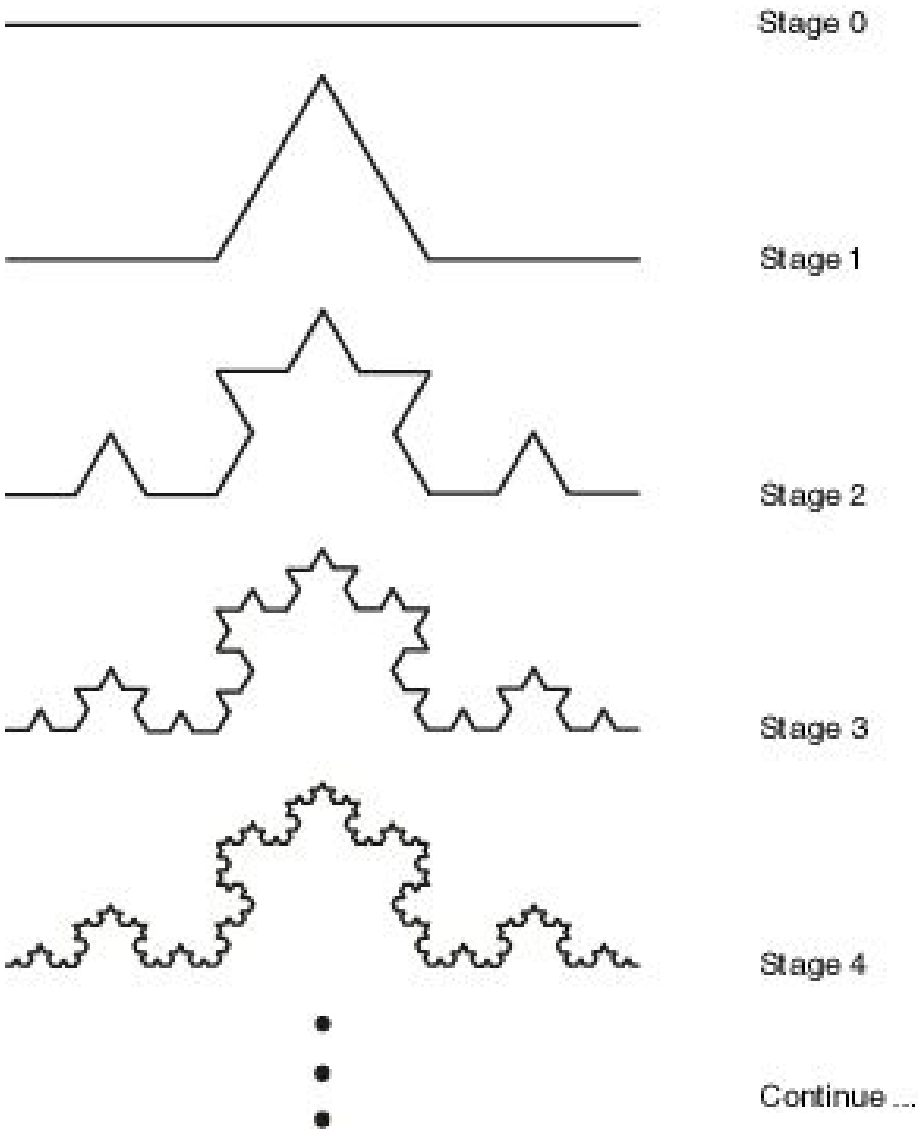
- Which points belong to the Cantor set?
  - We use base 3 notation, e.g., $(0.102)_3 = 1/3 + 0/3^2 + 2/3^3$
- Since we always remove the middle third of each interval, the points belonging to the Cantor set are those admitting at least one representation in base 3 with no digit "1"
  - E.g., 0.0202 belongs to the Cantor set, but 0.0221 does not
  - Note that 0.0222.. does belong to the Cantor set, even if 0.0222... = 0.1
- The Cantor set has uncountably many points
  - Proof: by diagonalization (similar to the proof that real numbers in [0,1] are uncountable)

Cantor set: summary

- The Cantor set has zero measure
- The Cantor set contains uncountable many points

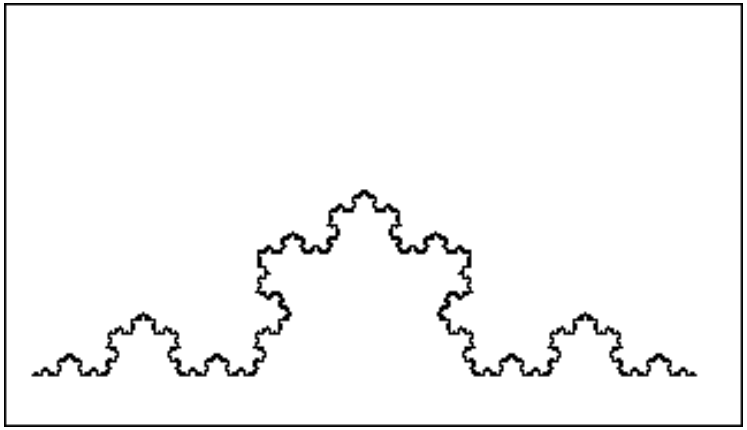
The Koch curve

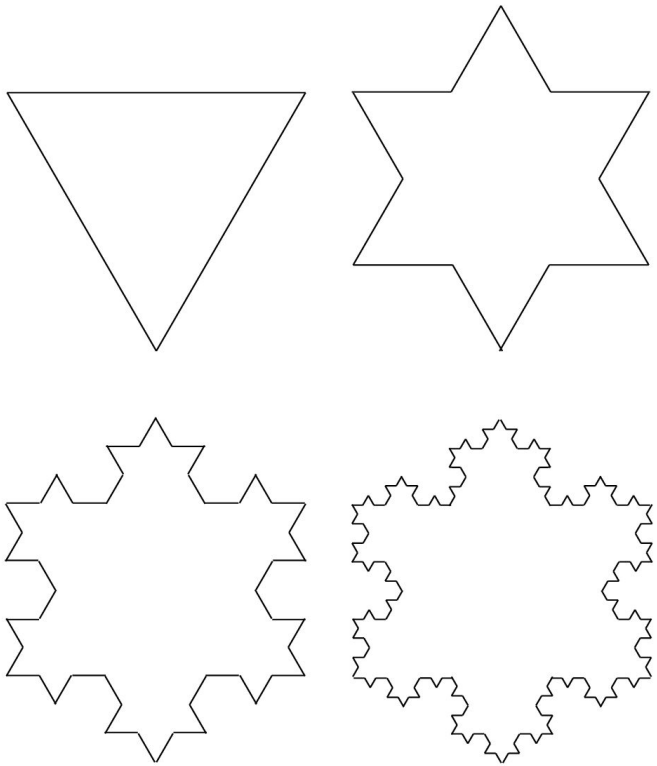
- Construction
  - Start with a line of unitary length
  - Remove the middle third, and replace id with two segments of length 1/3
  - Iterate
- At stage n the curve has 4^n segments of length $1/3^n$
  - Total length: $(4/3)^n$



The Koch curve

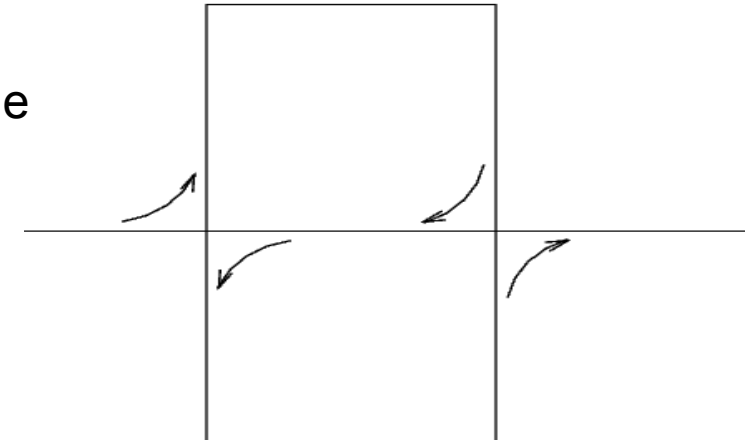
- The Koch curve does not admit a tangent at any point
  - the curve is made entirely of corners
- The Koch curve has infinite length
- Extension: *Koch snowflake*
  - Finite area, infinite perimeter

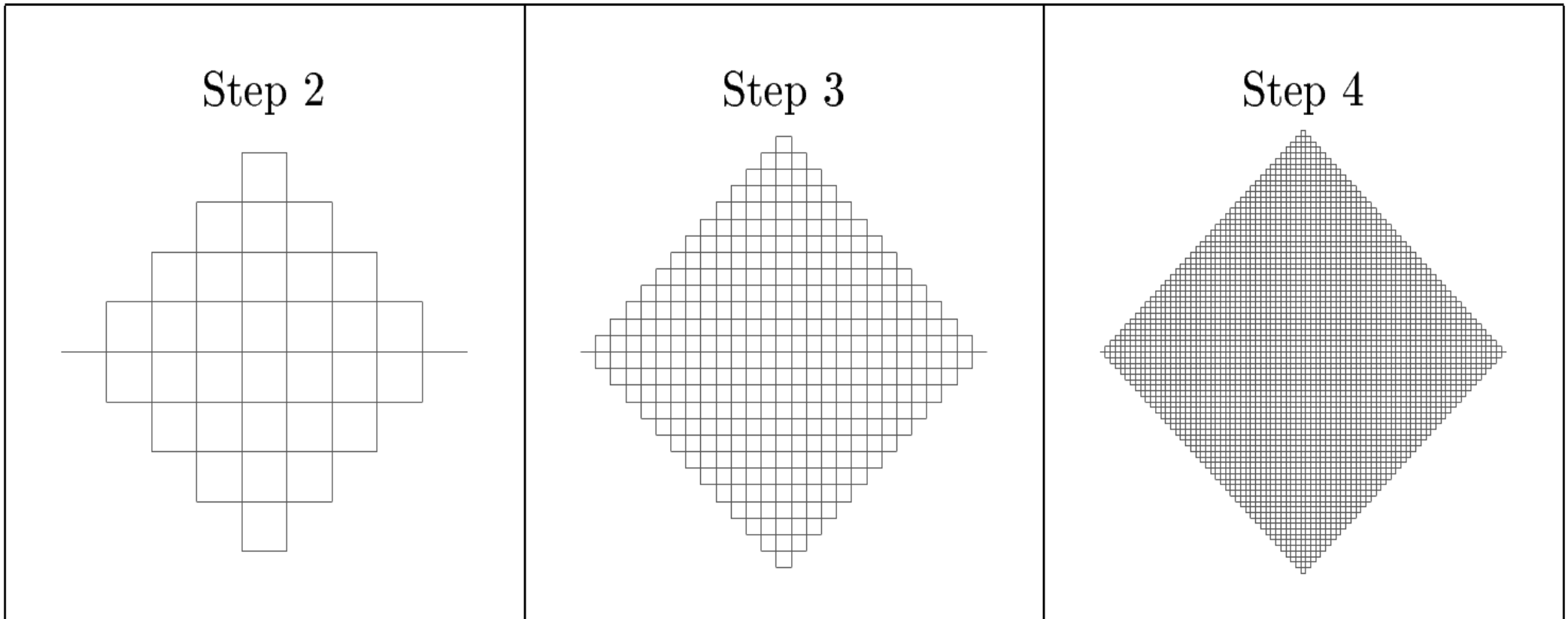




The Peano curve

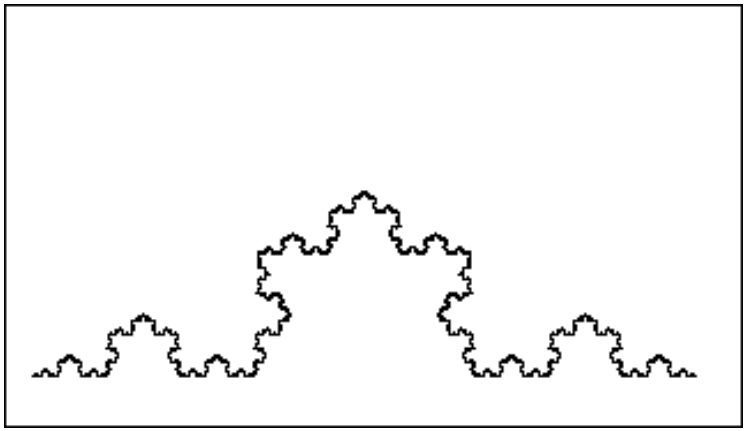
Single step of the Peano curve

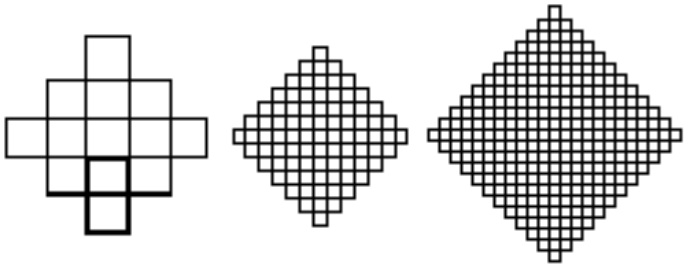




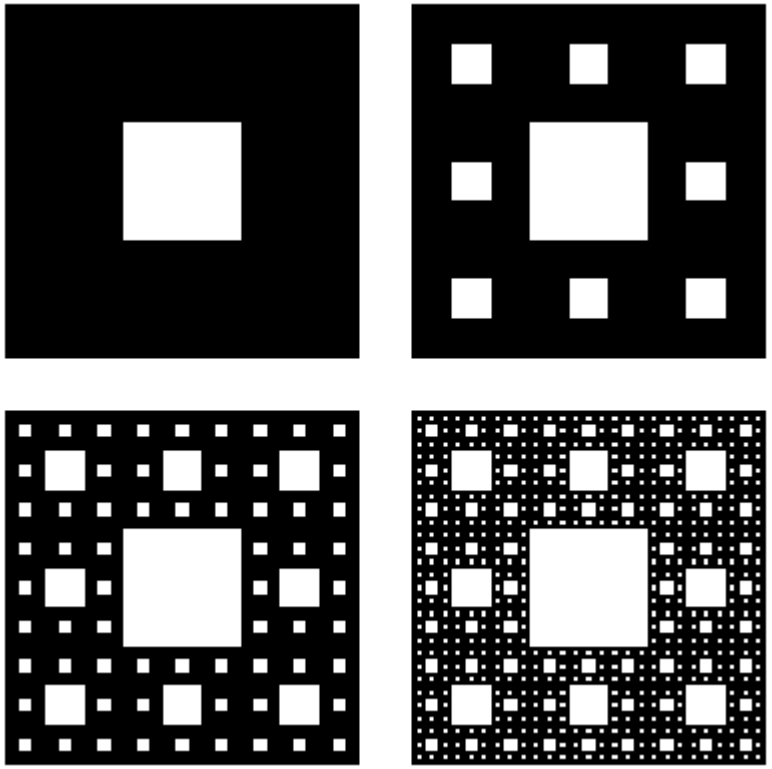
Recap

- Cantor set
  - Point-like (zero measure)
- Koch curve
  - Line-like (infinite length)
- Peano curve
  - Space-filling curve

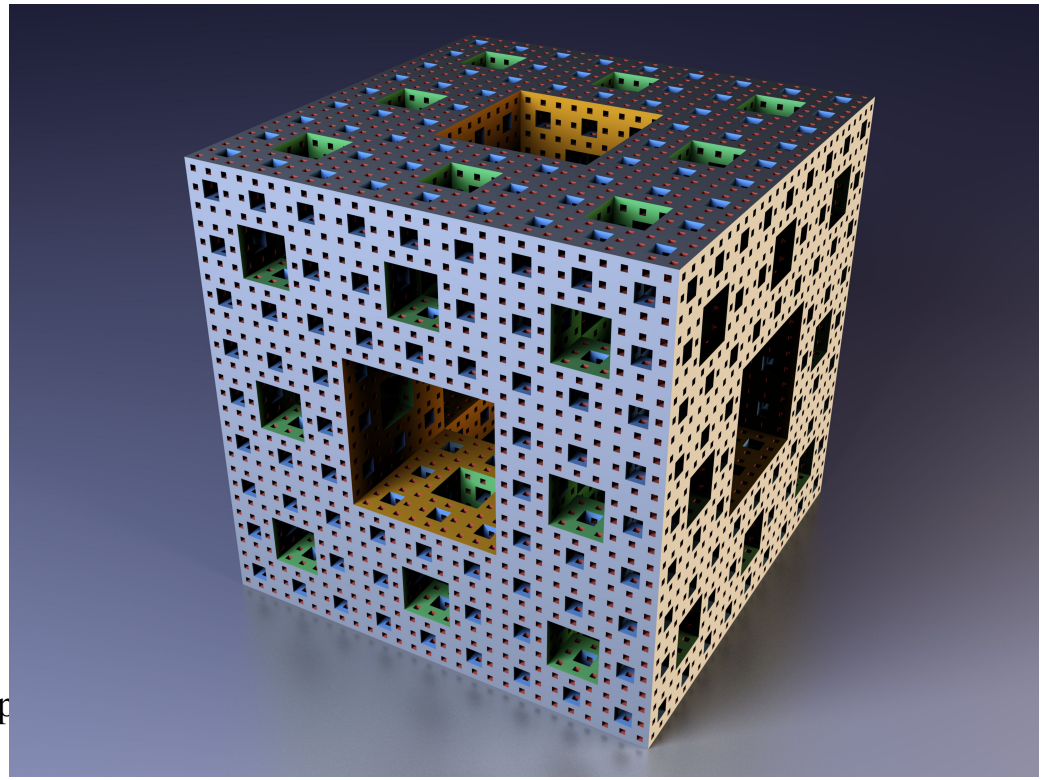




Other similar figures



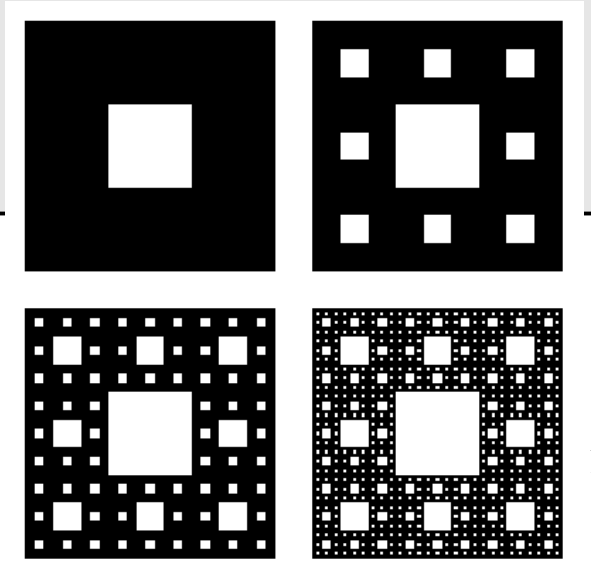
Sierpinski Carpet
(zero area)



Monger Sponge
(infinite surface
area, zero volume)

Drawing the Sierpinski carpet

```
/* Returns 1 iff pixel at coordinates (x,y) belongs to the
interior of the sierpinski carpet, and therefore the
corresponding pixel must be filled */
int sierpinski(int x, int y)
{
    while ( x>0 || y>0 ) {
        if ( x%3==1 && y%3==1 ) /* at center of square */
            return 0;
        x /= 3;
        y /= 3;
    }
/* If all square levels are checked and the pixel is not in the
center of any, it must be filled */
    return 1;
}
```



Adapted from http://en.wikipedia.org/wiki/Sierpinski_carpet

Fractal dimension

- Suppose that we have a stick of length *a*, and use it to measure an object. If the object is *N* units of length *a* each, its measure is $N \times a$
- If we reduce the length *a* of the stick, *N* grows

Line

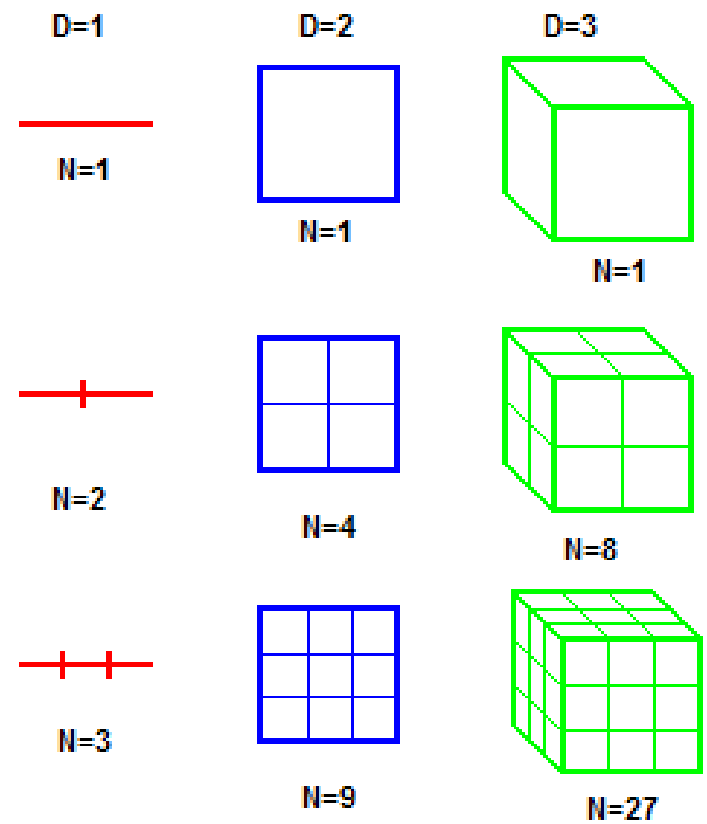
a	*N*
1	1
1 / 2	2
1 / 3	3
1 / 4	4

Square

a	*N*
1	1
1 / 2	4
1 / 3	9
1 / 4	16

Cube

a	*N*
1	1
1 / 2	8
1 / 3	27
1 / 4	64



$$N = \left(\frac{1}{a}\right)^D \rightarrow D = \frac{\log N}{\log \frac{1}{a}}$$

D = (fractal) dimension

Fractal dimension of the Cantor set

- N=1 sticks of length a=1
- N=2 sticks of length a=1 / 3
- N=4 sticks of length a=1 / 9

Cantor set

stage	*a*	*N*
0	1	1
1	1 / 3	2
2	1 / 9	4
…	…	…
n	$1 / 3^n$	2^n

$$D = \frac{\log N}{\log \frac{1}{a}} = \frac{\log 2^n}{\log 3^n} = \frac{\log 2}{\log 3} \approx 0.63$$

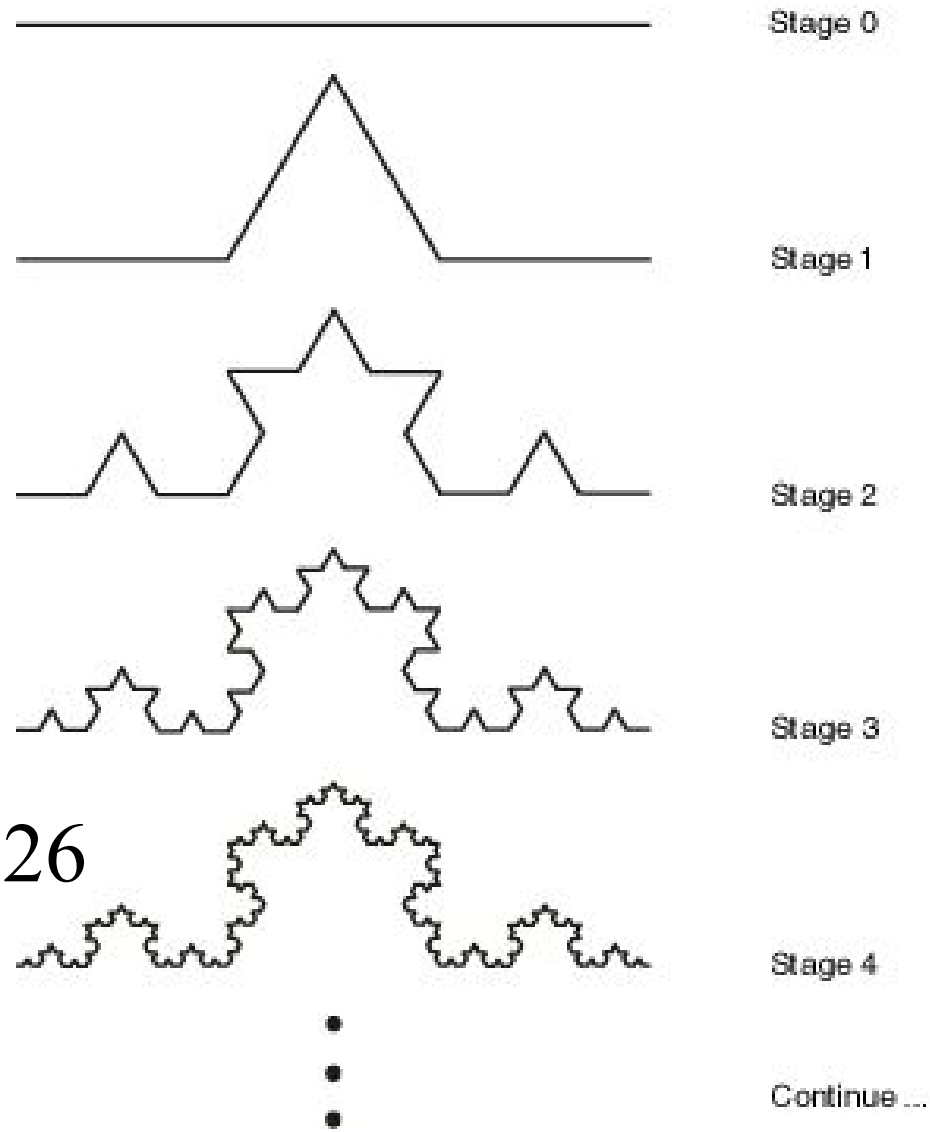
Fractal dimension of the Koch curve

- N=1 sticks of length a=1
- N=4 sticks of length a=1 / 3
- N=16 sticks of length a=1 / 9

Koch curve

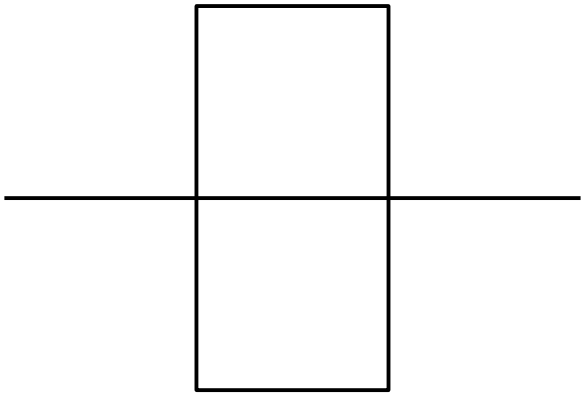
stage	*a*	*N*
0	1	1
1	1 / 3	4
2	1 / 9	16
…	…	…
n	$1 / 3^n$	4^n

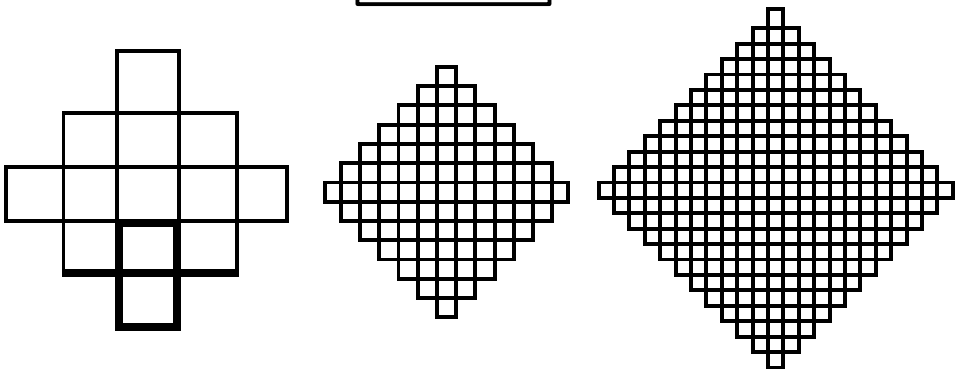
$$D = \frac{\log N}{\log \frac{1}{a}} = \frac{\log 4^n}{\log 3^n} = \frac{\log 4}{\log 3} \approx 1.26$$



Fractal dimension of the Peano curve

- N=1 sticks of length a=1
- N=9 sticks of length a=1 / 3
- N=81 sticks of length a=1 / 9





Peano curve

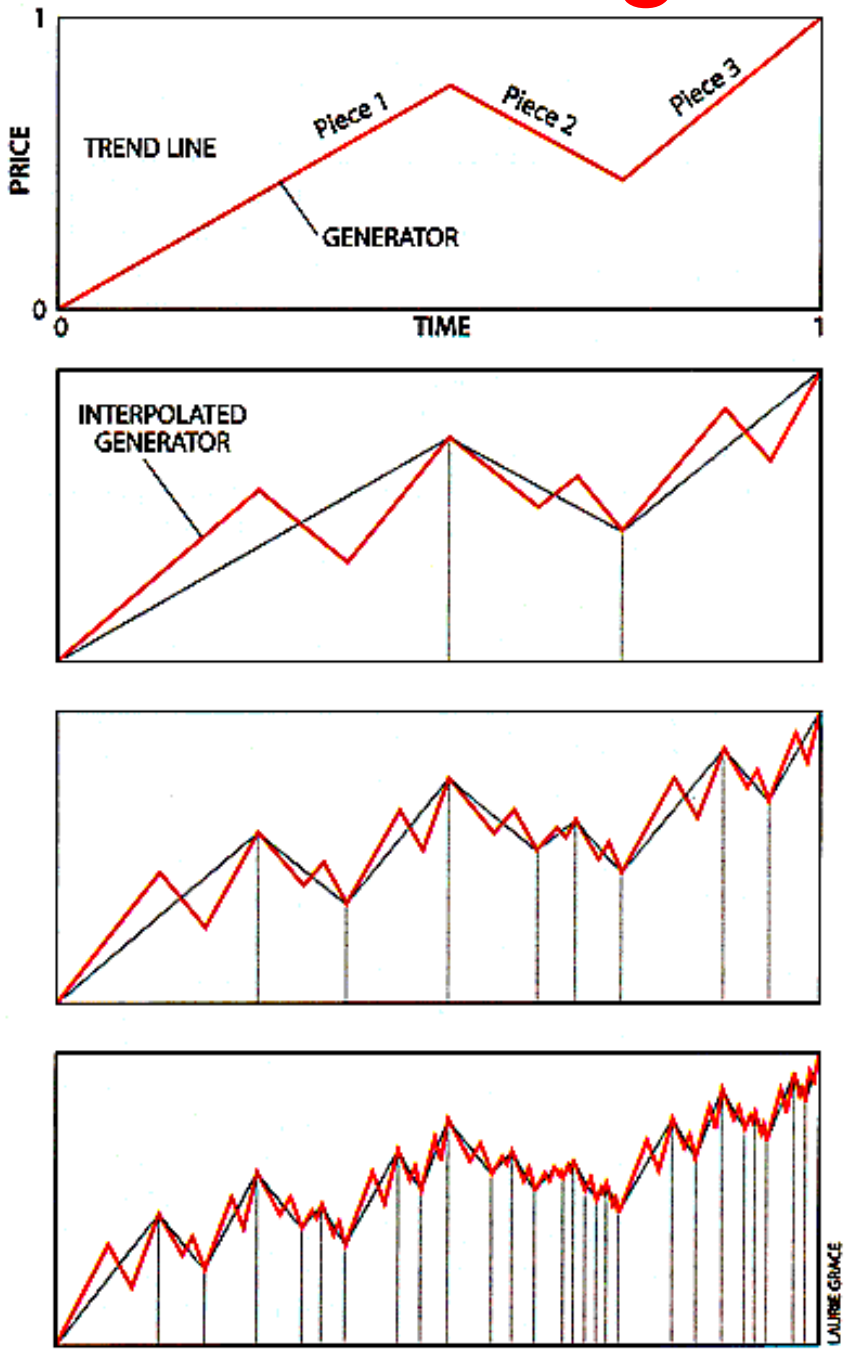
stage	*a*	*N*
0	1	1
1	1 / 3	9
2	1 / 9	81
…	…	…
n	$1 / 3^n$	9^n

$$D = \frac{\log N}{\log \frac{1}{a}} = \frac{\log 9^n}{\log 3^n} = \frac{\log 9}{\log 3} = 2$$

The Peano curve is a “line-like” object with the same dimension of a surface!

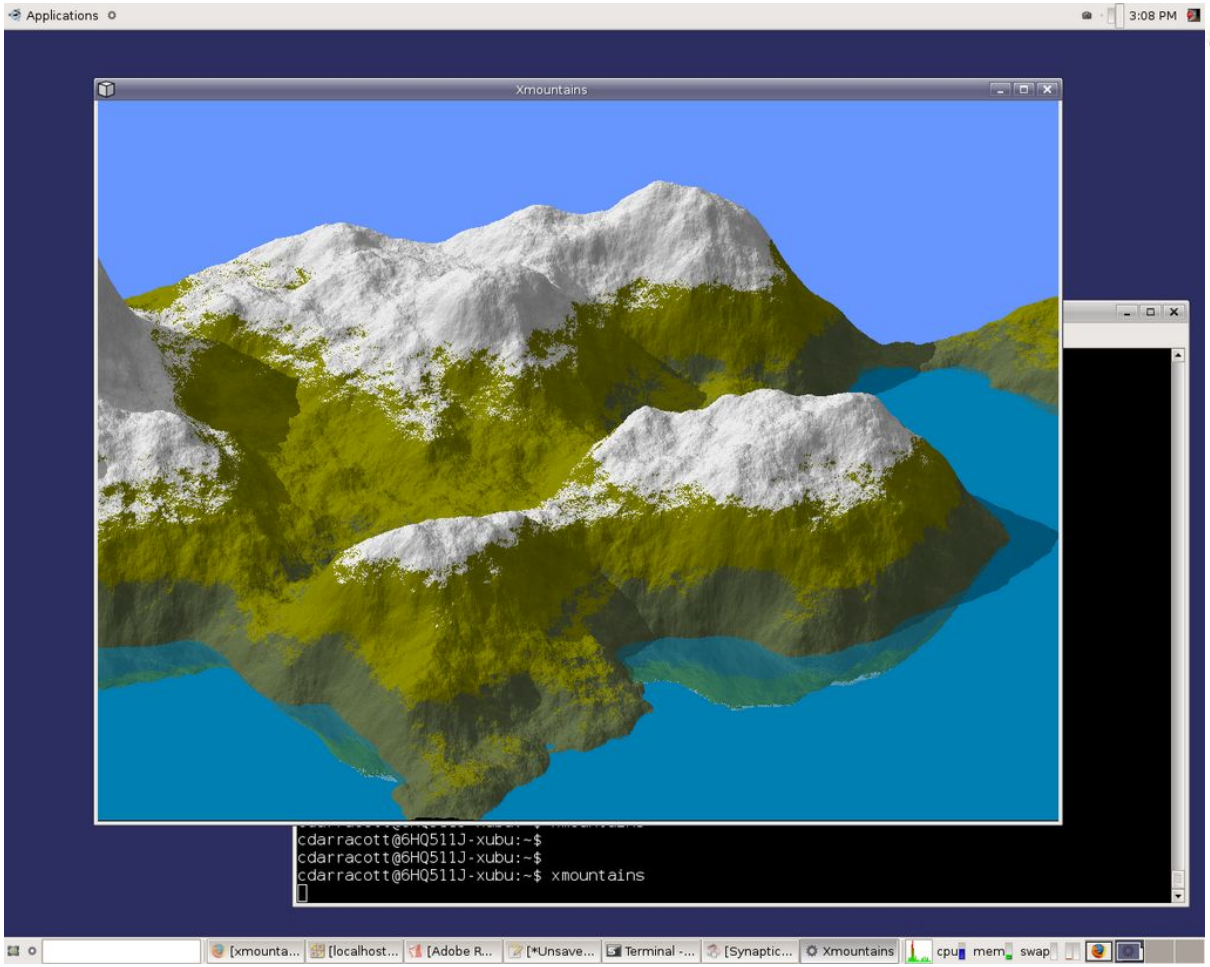
Self-similarity: stock market

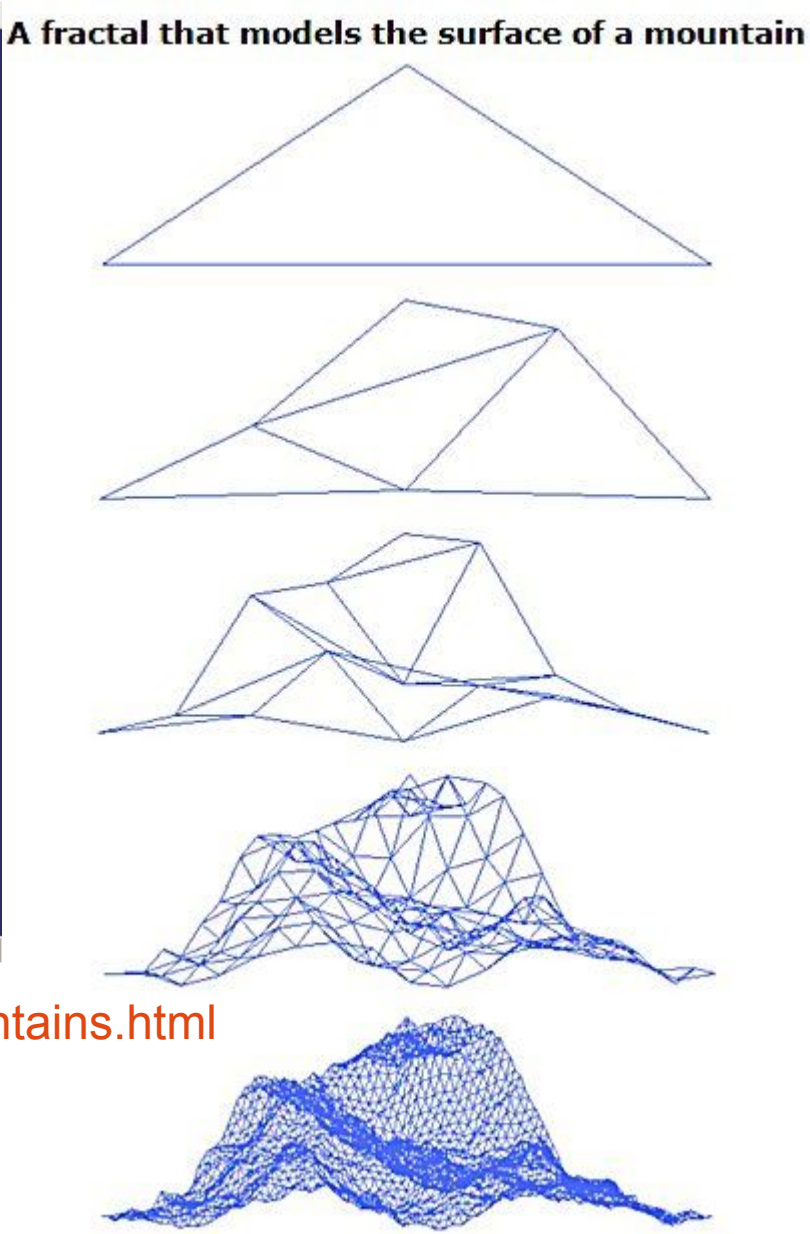
Recursive generator of stock prices



Benoit B. Mandelbrot, A Multifractal Walk Down Wall Street, Scientific, American, Feb 6, 1999

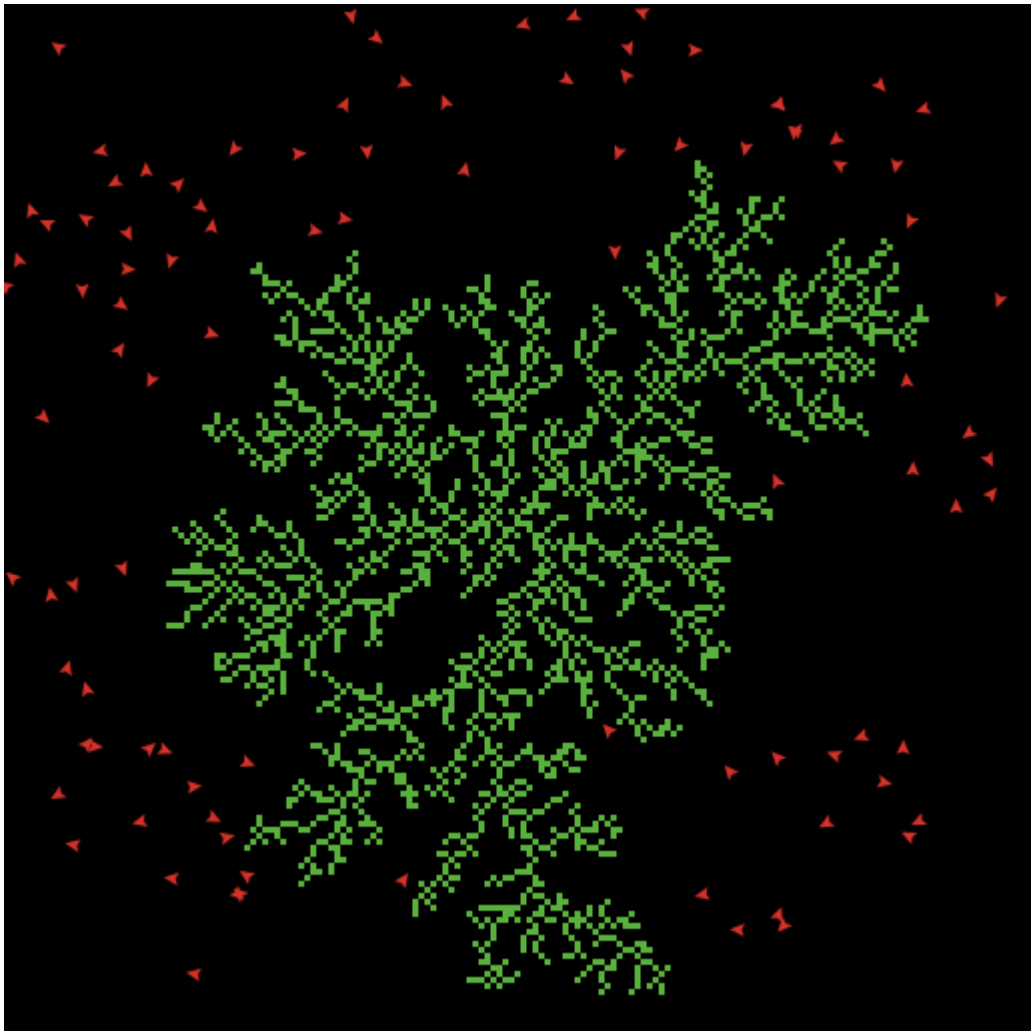
Example: mountains





http://www2.epcc.ed.ac.uk/~spb/xmountains/about_xmountains.html

Examples: diffusion limited aggregation



NetLogo: *Sample Models → Chemistry and Physics → Diffusion Limited Aggregation → DLA Simple*