Fractals

Moreno Marzolla
Dip. di Informatica—Scienza e Ingegneria (DISI)
Università di Bologna

http://www.moreno.marzolla.name/
Geometric Objects

• Man-made objects are geometrically simple (e.g., rectangles, circles, triangles...).
  – Those simple objects seem appropriate for describing many artifacts (e.g., this screen is roughly rectangular, tires are circular, power lines are long lines, etc.

• What about natural objects? (snowflake, tree, mountain, clouds...)
  – These objects have been considered exceptions to the rule
Fractals

- The term fractal was coined by the mathematician Benoît B. Mandelbrot to differentiate geometric figures with other figures that cannot be easily classified in conventional terms.

*Clouds are not spheres, mountains are not cones, coastlines are not circles, and bark is not smooth, nor does lightning travel in a straight line.*

—Mandelbrot, in his introduction to *The Fractal Geometry of Nature*
Fractals


Fractals

- **Self-similarity**
  - Fractals are self-similar at several scales, meaning that a small portion of a fractal looks similar to the whole object.

- **Fractional dimension**
  - Fractals have *fractional dimension*: a fractal object with dimension 1.5 is somewhat more than a line (dimension 1) and less than a plane (dimension 2).
The Cantor Set

- Start with the interval \([0, 1]\)
- Remove the middle third: \([0, 1/3] \cup [2/3, 1]\)
- Remove the middle third for each interval: \([0, 1/9] \cup [2/9, 1/3] \cup [2/3, 7/9] \cup [8/9, 1]\)
- Iterate the procedure \textit{forever}
Measure of the Cantor set

• We want to measure the “width” of the Cantor set
• At step $n$ (starting from $n = 0$) the Cantor set contains $2^n$ intervals of width $1/3^n$ each; total width is $(2/3)^n$
• For $n \to +\infty$ the total width becomes 0
Points in the Cantor set

- Which points belong to the Cantor set?
  - We use base 3 notation, e.g., \((0.102)_3 = 1/3 + 0/3^2 + 2/3^3\)

- Since we always remove the middle third of each interval, the points belonging to the Cantor set are those admitting at least one representation in base 3 with no digit “1”
  - E.g., 0.0202 belongs to the Cantor set, but 0.0221 does not
  - Note that 0.0222... \text{ does} belong to the Cantor set, even if 0.0222... = 0.1

- The Cantor set has uncountably many points
  - Proof: by diagonalization (similar to the proof that real numbers in \([0,1]\) are uncountable)
Cantor set: summary

• The Cantor set has zero measure
• The Cantor set contains uncountable many points
The Koch curve

- Construction
  - Start with a line of unitary length
  - Remove the middle third, and replace it with two segments of length $1/3$
  - Iterate

- At stage $n$ the curve has $4^n$ segments of length $1/3^n$
  - Total length: $(4/3)^n$
The Koch curve

- The Koch curve does not admit a tangent at any point
  - the curve is made entirely of corners
- The Koch curve has infinite length
- Extension: Koch snowflake
  - Finite area, infinite perimeter
The Peano curve

Single step of the Peano curve

Step 2

Step 3

Step 4
Recap

- Cantor set
  - Point-like (zero measure)
- Koch curve
  - Line-like (infinite length)
- Peano curve
  - Space-filling curve
Other similar figures

Sierpinski Carpet
(zero area)

Monger Sponge
(infinite surface area, zero volume)
/* Returns 1 iff pixel at coordinates (x,y) belongs to the interior of the sierpinski carpet, and therefore the corresponding pixel must be filled */

int sierpinski(int x, int y)
{
    while ( x>0 || y>0 ) {
        if ( x%3==1 && y%3==1 ) /* at center of square */
            return 0;
        x /= 3;
        y /= 3;
    }
    /* If all square levels are checked and the pixel is not in the center of any, it must be filled */
    return 1;
}

Adapted from http://en.wikipedia.org/wiki/Sierpinski_carpet
Fractal dimension

- Suppose that we have a stick of length $a$, and use it to measure an object. If the object is $N$ units of length $a$ each, its measure is $N \times a$.
- If we reduce the length $a$ of the stick, $N$ grows.

<table>
<thead>
<tr>
<th>Line</th>
<th>$a$</th>
<th>$N$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>1/2</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>1/3</td>
<td>3</td>
<td>1</td>
</tr>
<tr>
<td>1/4</td>
<td>4</td>
<td>1</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Square</th>
<th>$a$</th>
<th>$N$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>1/2</td>
<td>4</td>
<td>1</td>
</tr>
<tr>
<td>1/3</td>
<td>9</td>
<td>1</td>
</tr>
<tr>
<td>1/4</td>
<td>16</td>
<td>1</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Cube</th>
<th>$a$</th>
<th>$N$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>1/2</td>
<td>8</td>
<td>1</td>
</tr>
<tr>
<td>1/3</td>
<td>27</td>
<td>1</td>
</tr>
<tr>
<td>1/4</td>
<td>64</td>
<td>1</td>
</tr>
</tbody>
</table>

\[ N = \left( \frac{1}{a} \right)^D \rightarrow D = \frac{\log N}{\log \frac{1}{a}} \]

$D = \text{(fractal) dimension}$
Fractal dimension of the Cantor set

- $N=1$ sticks of length $a=1$
- $N=2$ sticks of length $a=1/3$
- $N=4$ sticks of length $a=1/9$

<table>
<thead>
<tr>
<th>stage</th>
<th>$a$</th>
<th>$N$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>$1/3$</td>
<td>2</td>
</tr>
<tr>
<td>2</td>
<td>$1/9$</td>
<td>4</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>n</td>
<td>$1/3^n$</td>
<td>$2^n$</td>
</tr>
</tbody>
</table>

\[
D = \frac{\log N}{\log \frac{1}{a}} = \frac{\log 2^n}{\log 3^n} = \frac{\log 2}{\log 3} \approx 0.63
\]
Fractal dimension of the Koch curve

- $N=1$ sticks of length $a=1$
- $N=4$ sticks of length $a=1/3$
- $N=16$ sticks of length $a=1/9$

Koch curve

<table>
<thead>
<tr>
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</tr>
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<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>1/3</td>
<td>4</td>
</tr>
<tr>
<td>2</td>
<td>1/9</td>
<td>16</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>n</td>
<td>1/3^n</td>
<td>4^n</td>
</tr>
</tbody>
</table>

$$D = \frac{\log N}{\log \frac{1}{a}} = \frac{\log 4^n}{\log 3^n} = \frac{\log 4}{\log 3} \approx 1.26$$
Fractal dimension of the Peano curve

- $N=1$ sticks of length $a=1$
- $N=9$ sticks of length $a=1/3$
- $N=81$ sticks of length $a=1/9$

<table>
<thead>
<tr>
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<th>$N$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>1/3</td>
<td>9</td>
</tr>
<tr>
<td>2</td>
<td>1/9</td>
<td>81</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>n</td>
<td>1/3$^n$</td>
<td>9$^n$</td>
</tr>
</tbody>
</table>

$$D = \frac{\log N}{\log a} = \frac{\log 9^n}{\log 1/3^n} = \frac{\log 9}{\log 3} = 2$$

The Peano curve is a “line-like” object with the same dimension of a surface!
Self-similarity: stock market

GOOG: 1 day

GOOG: 1 month

GOOG: 6 months

GOOG: 1 year
Recursive generator of stock prices

Example: mountains

A fractal that models the surface of a mountain

http://www2.epcc.ed.ac.uk/~spb/xmountains/about_xmountains.html
Examples: diffusion limited aggregation

NetLogo: Sample Models → Chemistry and Physics → Diffusion Limited Aggregation → DLA Simple